The remarkable Tameness of String Effective Actions

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Based on:

2112.06995 with Ben Bakker, Christian Schnell, Jacob Tsimerman

2112.08383 Tameness Conjecture

2206.00697 with Stefano Lanza, Chongchuo Li

220n.nnnn with Mike Douglas, Lorenz Schlechter

Introduction

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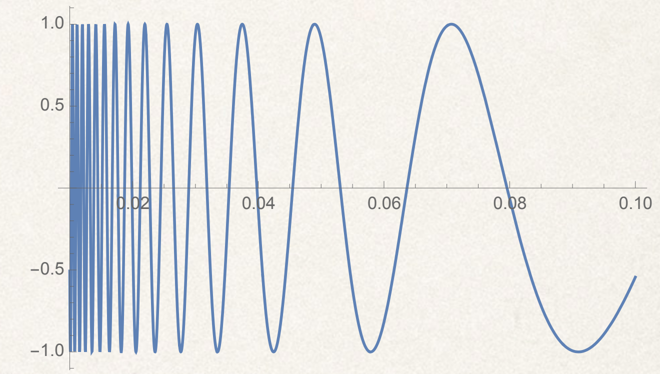

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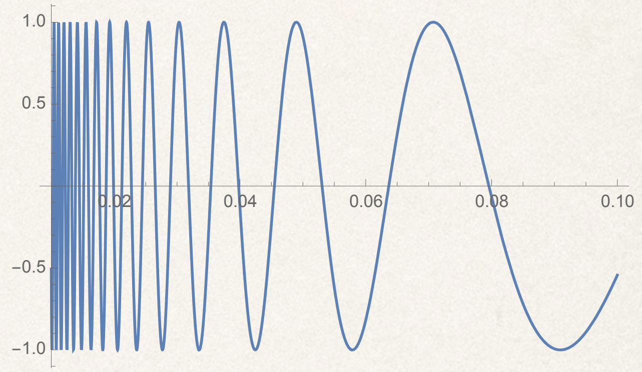

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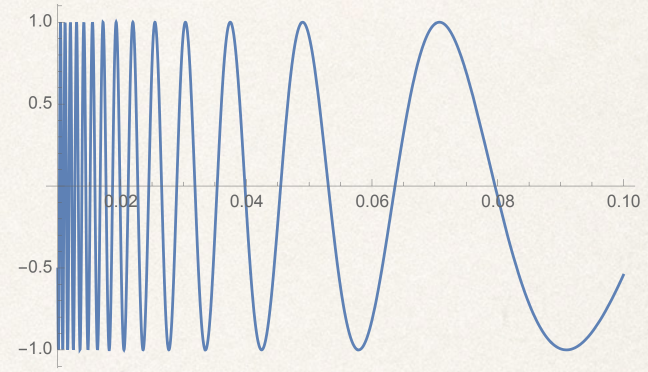

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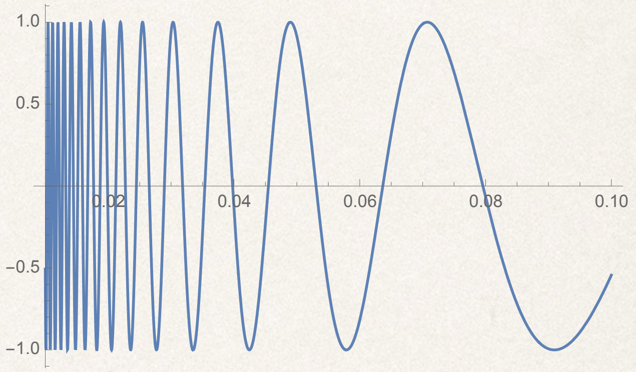

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Physics is more tame, isn't it?

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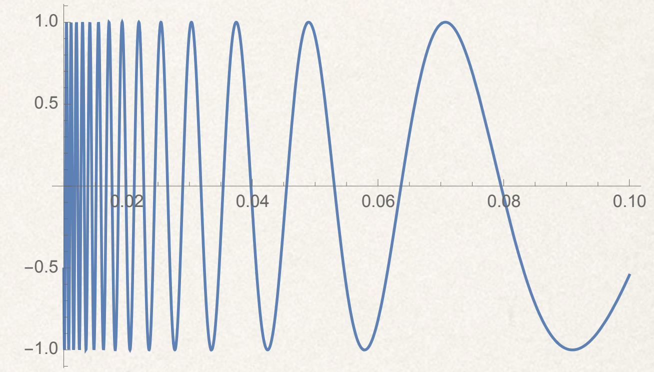

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What is a good Tameness Principle?

Finiteness as a tameness principle?

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Finiteness criterion seems to be a yes/no-criterion:
Can we turn finiteness into a structural criterion?

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> Tameness principle: demand that theories are formulated within 'Tame geometry' or 'o-minimal geometry'

(needed in the proof of [Bakker,TG,Schnell,Tsimerman])

Rough tameness statements

(1) Observe that effective theories derived from string theory that are valid below a fixed finite energy scale have tame coupling functions, field spaces, and parameter spaces.

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(2) Tame effective theories/QFTs remain tame when including perturbative corrections up to a fixed loop level.

Effective theories

- Effective theory:

$$\mathcal{L} = \frac{1}{2}R - g_{ij}(\phi, \lambda)\, D_\mu \phi^i D^\nu \phi^j - f_{\alpha\beta}(\phi, \lambda)\mathrm{tr}(F^\alpha_{\mu\nu}(F^\beta)^{\mu\nu}) - V(\phi, \lambda) + \ldots$$

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special 'tame' set — special 'tame' function

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Tame Geometry

A brief introduction to o-minimal structures

A mathematical structure with finiteness

- Geometry: develop a mathematical framework for geometers:
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  - built theory with polynomial equalities and inequalities over $\mathbb{R}$ (with ordering "$>$") that has only decidable statements [Tarski]
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- Resulting picture:
  → o-minimal structures define a tame topology
  → give a generalization of real algebraic geometry
  → strong finiteness properties   intro book [van den Dries]

Recent lectures: Jacob Tsimerman (2021 Princeton lectures, 2022 Fields institute)

Tame topology: o-minimal structures

- Basic idea: specify collection S_n of tame sets $\mathcal{A} \subset \mathbb{R}^n$ and allowed tame functions $f : \mathbb{R}^n \to \mathbb{R}^m$

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- O-minimal structure (a 'tame structure'):

> Tameness assumption: tame sets S_1 of $\mathbb{R}$ are finite unions of intervals and points

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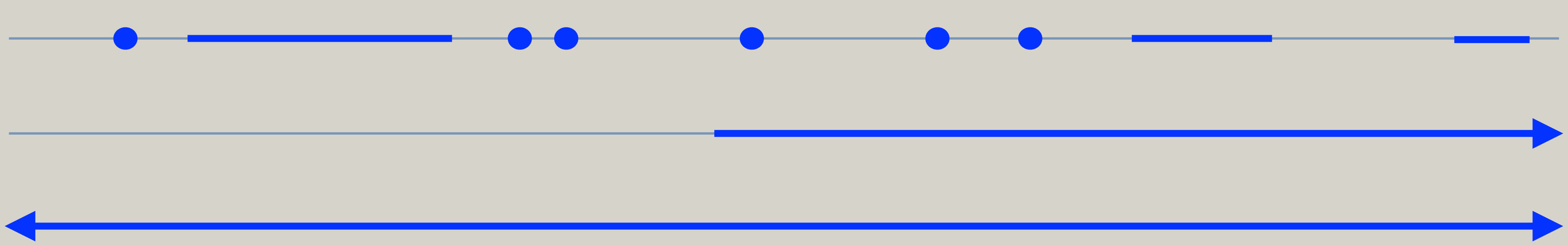
Tameness assumption: tame sets in $\mathbb{R}$



finitely many points/intervals

infinitely long intervals

whole real line

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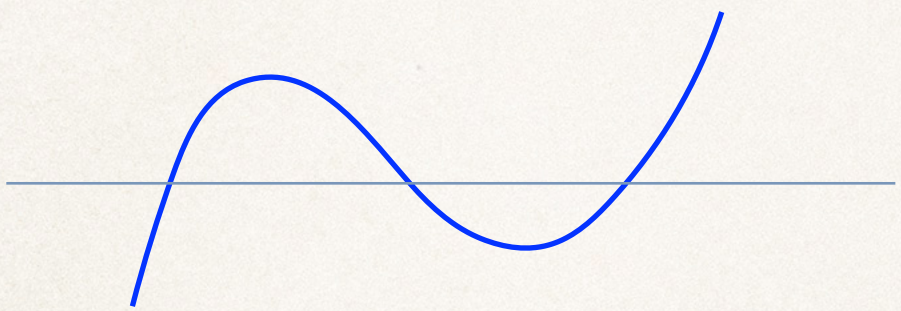
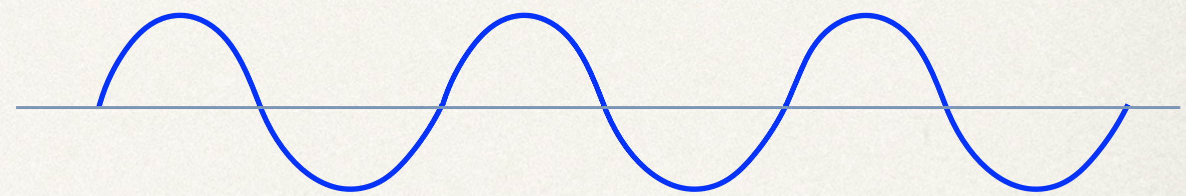
Example: polynomial function



Non-Example: $\sin(x), x \in \mathbb{R}$
is never a tame function



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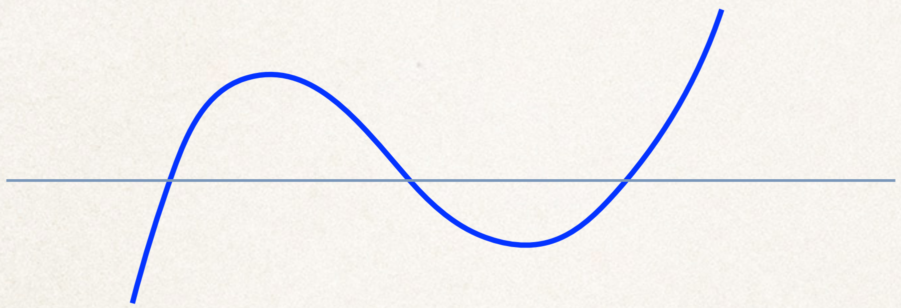

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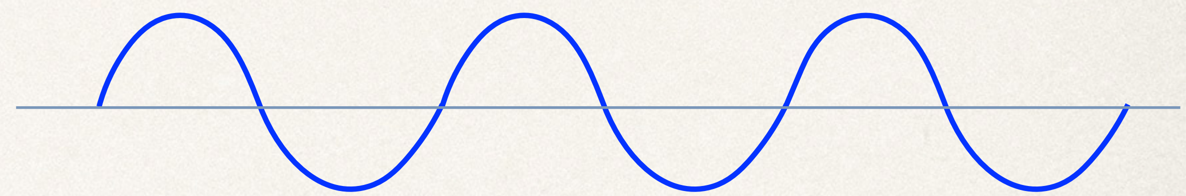
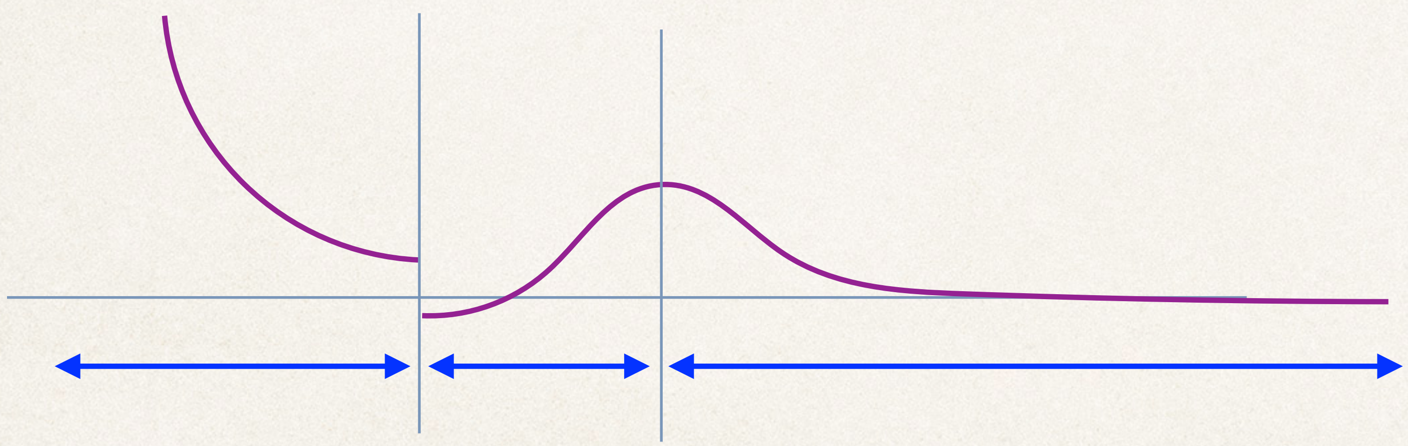


Theorem: tame $f : \mathbb{R} \to \mathbb{R}$



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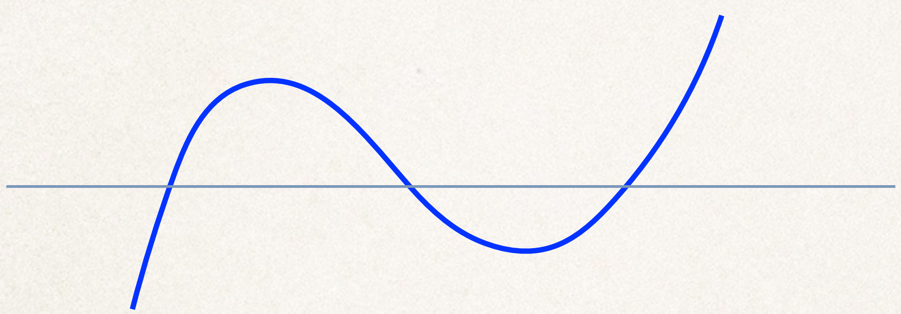

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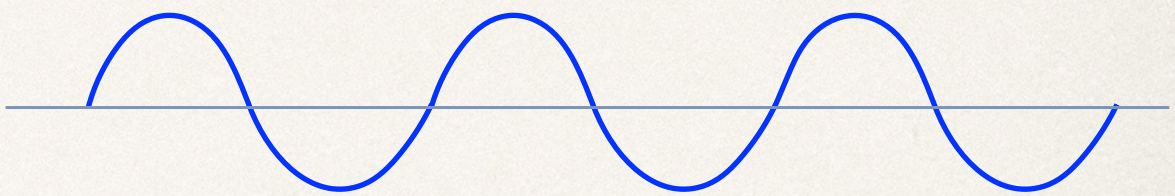

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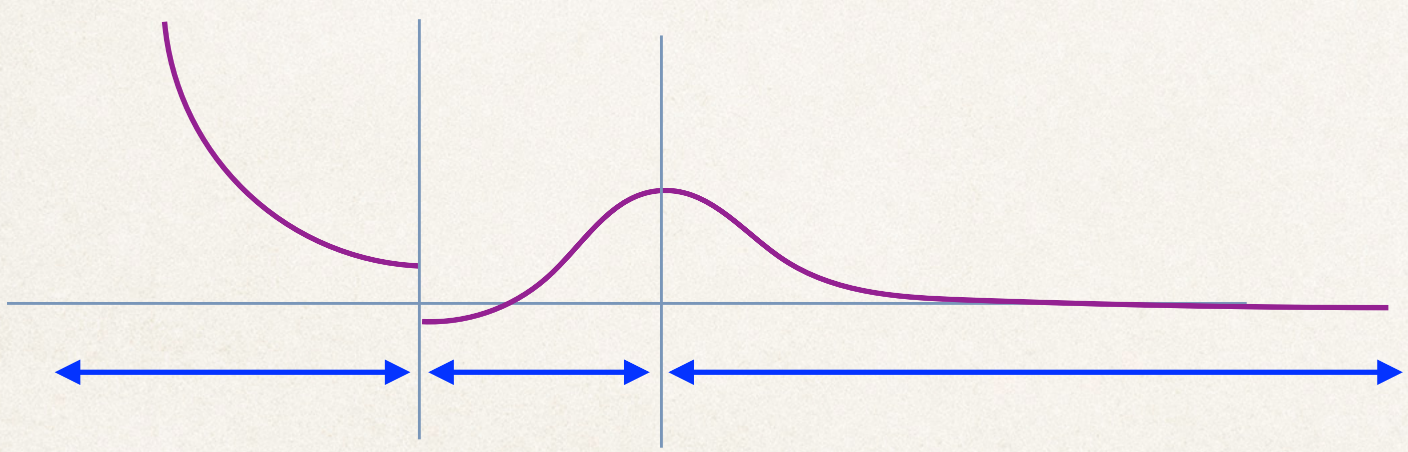

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(1) finitely many minima and maxima; (2) tame tail to infinity

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  - $\mathbb{R}_{\mathrm{exp}}$ plus restricted real analytic functions: $\mathbb{R}_{\mathrm{an,exp}}$ [van den Dries, Miller '94]

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polynomial — exponential

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needed for tameness of complex exponential: $e^z = e^r(\cos(\phi) + i\sin(\phi))$ $\quad 0 \leq \phi \leq c$

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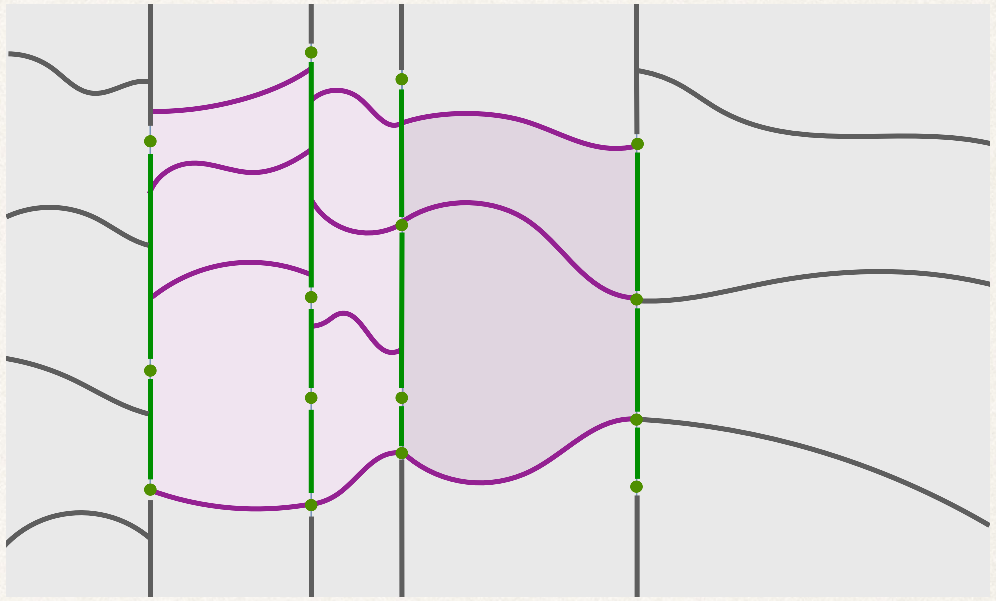
polynomial — exponential — restricted analytic

not tame in $\mathbb{R}_{\text{an,exp}}$: $\Gamma(x)$ on $(0, \infty)$; $\zeta(x)$ on $(1, \infty)$; error function

There is much more to say:

- Higher-dimensional tame functions and sets well understood
  - exists finite cell decomposition



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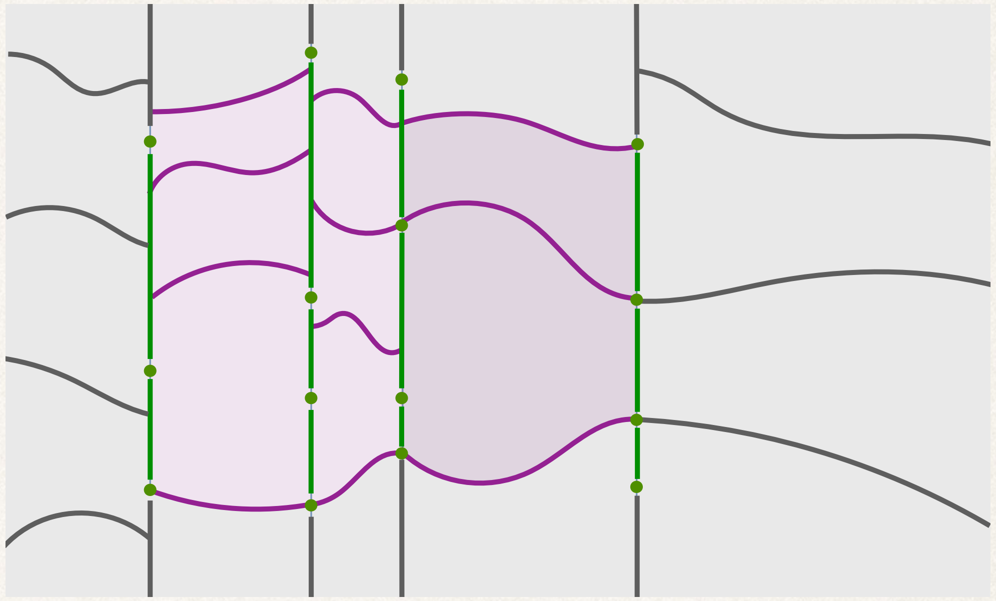

- Tameness used in many recent proofs of deep mathematics conjectures:
  - Ax-Schanuel conjecture for Hodge structures [Bakker,Tsimerman '17]
  - Griffiths' conjecture [Bakker,Brunebarbe,Tsimerman '18]
  - André-Oort conjecture [Pila,Shankar,Tsimerman '21]

  → very active field connecting logic, number theory, and geometry

Tameness at work

- Tameness statement: field space $\mathcal{M}$ tame manifold
  potential $V(\phi_1, \phi_2)$ tame function

- Integrate out heavy ϕ_1: $\mathcal{M}_{\text{vac}} = \left\{ \frac{\partial V}{\partial \phi_1} = 0 \right\} \cap \mathcal{M}$ → intersection of tame spaces

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$$V(\phi_1, \phi_2) \to V(\langle \phi_1 \rangle, \phi_2) \quad \rightarrow \text{ projection of tame function}$$

→ tameness classically preserved when lowering cut-off

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Tameness preserved at quantum level?

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Tameness preserved at quantum level?

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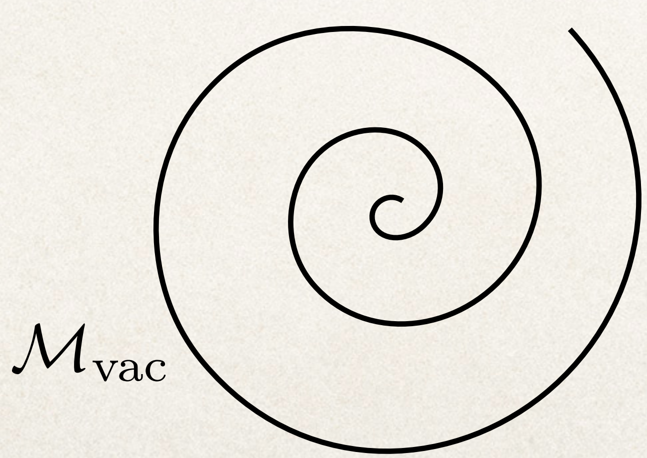

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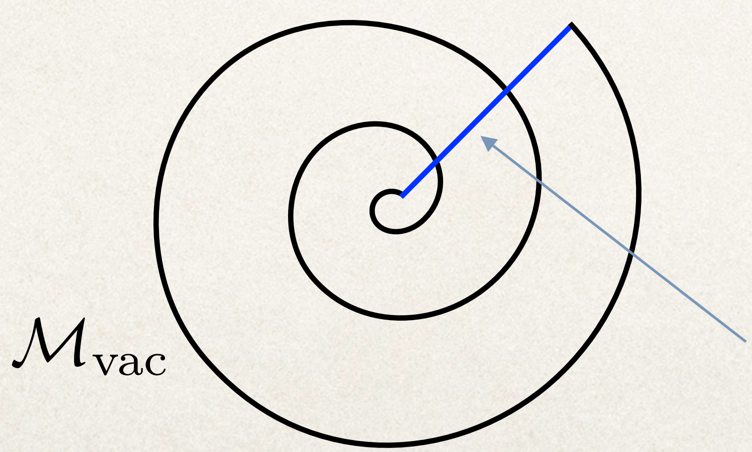

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→ cannot be tame, V not definable

linear project with infinitely many points

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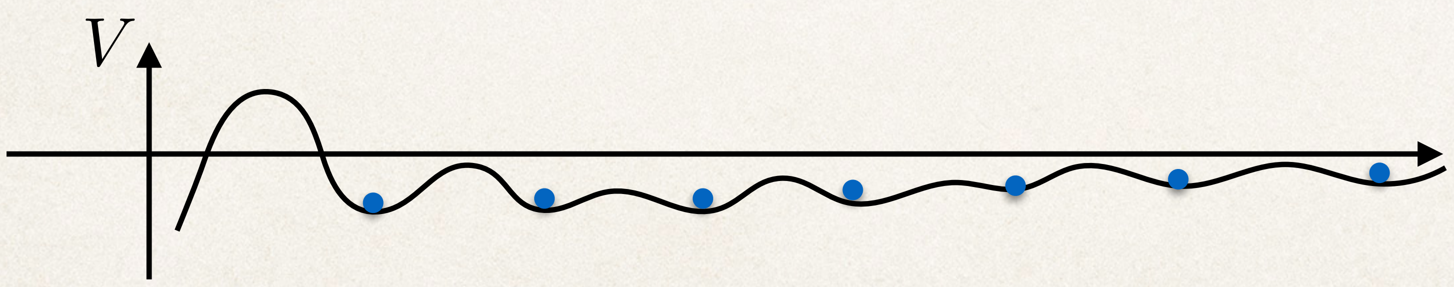

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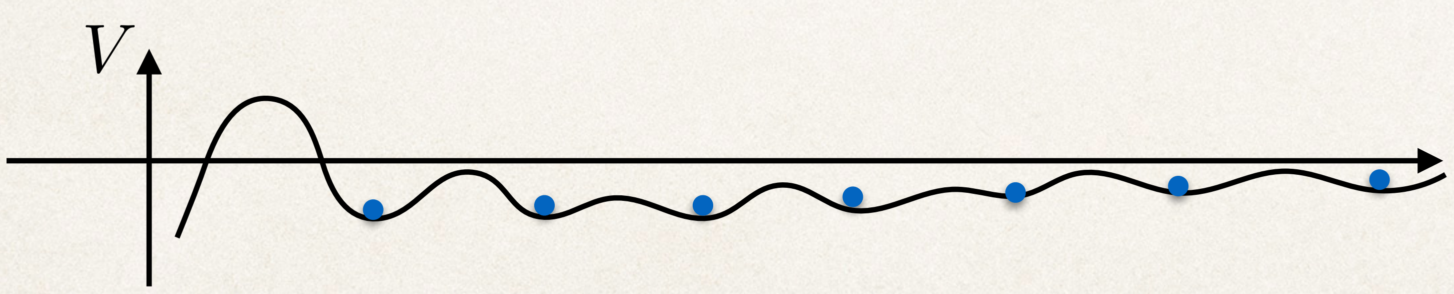

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many functions do not appear: $V(\phi) = \sin(\phi^{-1})$ $V(\phi) = \phi^8 \sin(\phi^{-1})$

→ no accumulation points of vacua

discussed by [Acharya,Douglas]

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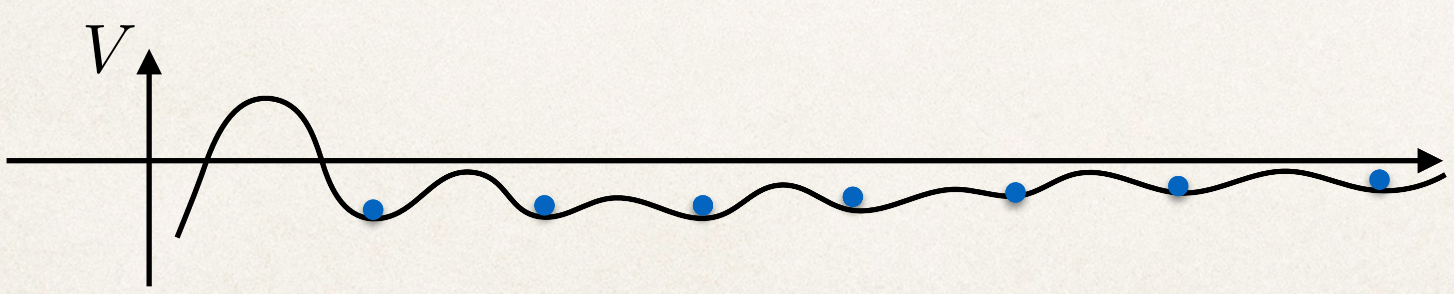

infinitely many vacua not compatible with tameness

recent suggestion by [Tachikawa] of QFT with scalar potential and undecidable statements is not tame

Tameness Conjecture

A new swampland conjecture

[TG] '21

Tameness conjecture:

All effective theories valid below a fixed finite energy cut-off scale that can be consistently coupled to quantum gravity are labelled by a tame parameter space and must have scalar field spaces and coupling functions that are tame in an o-minimal structure.

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Refined version:

The relevant o-minimal structure is $\mathbb{R}_{\mathrm{an,exp}}$.

Evidence for Tameness:

Supersymmetry + Strings

Tameness and supersymmetry: $\mathbf{N > 2}$

- Supergravity theories with N>2 supersymmetry in D≧4:

(1) scalar field spaces:

$$\mathcal{M} = \Gamma \backslash G / K$$

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(3) parameter spaces: are they tame?

check: sectrum/group ranks (e.g. choices for Γ, G) are finite in string compactifications → discrete infinite sets are never definable

Tameness in compactifications: **N=2**

- Less supersymmetry: N=2 compactifications on Calabi-Yau threefolds

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- Also recently shown:

  (a) Hodge star on $H^D(Y_D, \mathbb{C})$, period map are tame in $\mathbb{R}_{\mathrm{an,exp}}$

  [Bakker,Klingler,Tsimerman] '18

  (b) period integrals themselves are tame in $\mathbb{R}_{\mathrm{an,exp}}$

  [Bakker,Mullane] '22 + [Bakker,Tsimerman] to appear

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- Note: period integrals have 'parameters' (e.g. mirror intersection numbers)

  → non-trivial $\mathcal{P}$: would need finiteness of Calabi-Yau manifolds

Some relations to other conjectures

- Consider (non-trivial) function with discrete symmetry:

  $f(x) = f(x+n)\ , \quad n \in \mathbb{Z} \quad \Longrightarrow$ not tame for $x \in \mathbb{R}$

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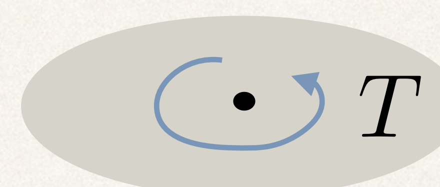
$$\vec{\Pi}(ze^{2\pi i}) = T \cdot \vec{\Pi}(z)$$



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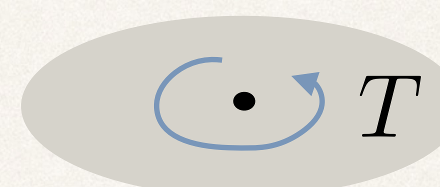

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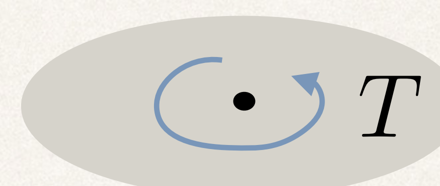

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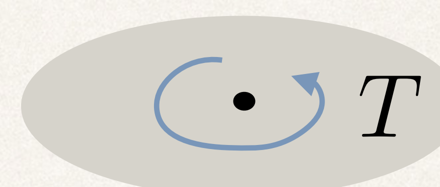

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Tameness + Distance conjecture = ♡ [TG,Lanza,Li]

$\rightarrow$ Talk by Stefano Lanza

Tameness in flux compactifications: N=0,1

- Type IIB/F-theory flux compactifications review: [Graña] [Kachru,Douglas] …

background flux: $G_4 \in H^4(Y_4, \mathbb{Z})$ $\int_{Y_4} G_4 \wedge G_4 = \ell$

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- fix G_4: scalar potential $V(z, \bar{z})$ is tame in $\mathbb{R}_{\text{an,exp}}$

→ finitely many minimum loci [Bakker,TG,Schnell,Tsimerman] '21

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- tameness (and finiteness) of locus of self-dual fluxes now part of general theorem

[Bakker,TG,Schnell,Tsimerman] '21

Evidence for Tameness:

Perturbative QFT

Tameness at quantum level

- General local QFT (renormalizable / EFT with cutoff)

ℓ -loop amplitude $\mathcal{A}_\ell(p, m)$

independent external momenta p_i

masses of particles in the loop m_α

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space of momenta (→ $\mathfrak{M}$)

parameters: masses, vertices (→ $\mathfrak{P}$)

- show that maps $\mathcal{A}_\ell$ are tame in $\mathbb{R}_{\text{an,exp}}$

[Douglas,TG,Schlechter] in preparation

→ Talk by Lorenz Schlechter

Tameness at quantum level

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$$I(p,m) = \int \left(\prod_{r=1}^{L} \frac{\mathrm{d}^d k}{i\pi^{d/2}}\right)\left(\prod_{j=1}^{n} \frac{1}{D_j^{v_j}}\right) \longrightarrow I(z) = \int_\gamma \Omega$$

review book by [Weinzierl]

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- Use: period integrals are tame maps in $\mathbb{R}_{\text{an,exp}}$

  [Bakker,Mullane] '22 + [Bakker,Tsimerman] to appear

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- Tame geometry and o-minimal structures are omnipresent in effective field theories arising from string theory

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- Combine other conjectures with Tameness Conjecture:

  tameness conjecture + distance conjecture [TG,Lanza,Li]
  tameness conjecture + swampland conjectures ? [TG,Lanza,van Vliet] in progress

Thanks!